Sharkovsky's Theorem and Its Converse: Constructing Continuous Maps of Prescribed Period

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Abstract

Sharkovsky's theorem asserts that the set of periods of a continuous map $f : I \to I$ of an interval into itself is completely determined by a single linear ordering of the natural numbers—a fact that, on first encounter, feels like it should not be true. This paper gives a self-contained exposition of both the forcing theorem and its converse. Most of the paper is spent on an explicit, constructive proof of the converse following Elaydi (1996), which I found the most illuminating path into the subject: given any positive integer r, we build a concrete piecewise-linear map f_r that has a period-r point but avoids every period preceding r in the Sharkovsky order. Along the way the natural direct proof of the forcing theorem due to Burns and Hasselblatt is sketched, the connection to Li and Yorke's "period three implies chaos" result is discussed, and a few applications to population dynamics and control are noted.

1 Introduction

In discrete dynamics one studies what happens when a single map $f : X \to X$ is iterated. When X is an interval in $\mathbb{R}$ and f is continuous, the set of possible periods turns out to be far more constrained than the simplicity of the setup would suggest; see [5] for a broader introduction. The foundational observation is due to Alexander Sharkovsky, who proved in 1964 [1] that the natural numbers can be linearly ordered so that the existence of a periodic orbit of one period forces the existence of orbits of every later period in that ordering.

Sharkovsky's paper, written in Ukrainian and published in a Soviet journal, went essentially unnoticed in the West for over a decade. In 1975, Li and Yorke [2] independently rediscovered the special case "period three implies every period," and in the same paper coined the term *chaos* in a precise mathematical sense. It was only afterward that the Western community recognized Sharkovsky's original result—which, it turned out, was strictly stronger.

The question that motivated this project is the converse: if you decide in advance which periods you want a map to have, can you actually build one? The answer is yes—Sharkovsky himself proved it—but the *how* is what I find most interesting. Elaydi [4] gives a construction that is explicit enough to compute by hand, and that is the approach I follow here.

After fixing notation and the two IVT lemmas that carry all the analytic weight (Section 2), I define the Sharkovsky ordering (Section 3), sketch the forcing theorem via Burns and Hasselblatt [3] (Section 4), and place Li and Yorke in context (Section 5). The rest of the paper is the converse (Section 6), which is where most of the interesting work is.

Sharkovsky's theorem is sometimes presented as if it requires substantial machinery, but the proof reduces to the Intermediate Value Theorem plus combinatorics. Keeping that visible throughout seemed like the right way to write this up, since it is part of what makes the theorem surprising.

2 Preliminaries

Throughout the paper $I \subseteq \mathbb{R}$ denotes a (possibly unbounded) interval and $f : I \to I$ a continuous function. We write $f^n = f \circ f \circ \cdots \circ f$ (n times) for the n-fold iterate.

Definition 1. A point $x \in I$ is periodic if $f^m(x) = x$ for some $m \in \mathbb{Z}_+$. The smallest such m is the prime period (or just period) of x, and we call m a period for f. The set $\mathcal{O}(x) = \{x, f(x), f^2(x), \ldots, f^{m-1}(x)\}$ is the cycle; we call x has period m and $\mathcal{O}(x)$ an m-cycle.

A fixed point is a periodic point of period 1. Note that if $f^n(x) = x$, then the prime period of x is a divisor of n.

The following two lemmas, both simple consequences of the Intermediate Value Theorem (IVT), constitute essentially the entire analytic input to Sharkovsky's theorem. The rest is combinatorics.

Lemma 2 (Fixed-point lemma)**.** *If $J = [a, b] \subseteq I$ is a closed interval and $f(J) \supseteq J$, then f has a fixed point in J.*

Proof. By hypothesis there exist $\alpha, \beta \in J$ with $f(\alpha) = a$ and $f(\beta) = b$. Then $f(\alpha) - \alpha \leq 0 \leq f(\beta) - \beta$, and by IVT the continuous function $g(x) = f(x) - x$ has a zero in J. □

This lemma is used constantly below, often without explicit mention. The covering condition $f(J) \supseteq J$ is easy to verify in our piecewise-linear examples just by checking endpoint values.

Definition 3. We say an interval J covers K, written $J \to K$, if $f(J) \supseteq K$.

Lemma 4 (Itinerary lemma)**.** *Let $J_0, J_1, \ldots, J_{n-1}$ be closed subintervals of I and suppose*

$$J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0.$$

Then there is a point $x \in J_0$ with $f^i(x) \in J_i$ for $0 \leq i < n$ and $f^n(x) = x$.

Proof. Take any K, J closed subintervals of I; we say $K \rightarrowtail J$ if $f(K) = J$. We claim that given any I, J closed subintervals where $I \to J$, there exists $K \subset I$, also closed subinterval, such that $K \rightarrowtail J$.

Thus, there exists a closed bound $K_{n-1} \subset J_{n-1}$ such that $K_{n-1} \rightarrowtail J_0$. Then, since $J_{n-1} \to K_{n-1}$, there exists closed interval $K_{n-2} \subset J_{n-2}$ such that $K_{n-2} \rightarrowtail K_{n-1}$. Inductively, there are closed real intervals $K_i \subset J_i$ for every integer $0 \leq i \leq n-1$. That is,

$$K_0 \rightarrowtail K_1 \rightarrowtail \cdots \rightarrowtail K_{n-1} \rightarrowtail J_0.$$

That is, $f^n(J_0) = K_0$. Since $K_0 \subset J_0$, by *Lemma 2* we say that there exists a point $x \in K_0$ such that x is the fixed point of f^n.

□

Such a cyclic sequence of coverings is called a loop. The point x produced by Lemma 4 is said to follow the loop, but its period need not be exactly n: it could be a proper divisor. To guarantee the exact period, one restricts to "elementary" loops in the sense of Burns and Hasselblatt.

Definition 5. A loop $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$ is elementary if every point that follows it has period exactly n (with respect to f).

Elementary loops produce periodic points of exactly the prescribed period, and can be detected combinatorially from an existing cycle without knowing f in the interior at all.

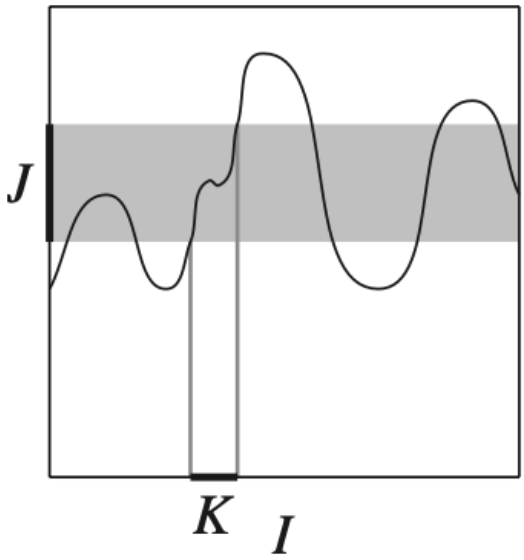

Figure 1: Finding $K \rightarrowtail J$.

3 The Sharkovsky Ordering

Definition 6 (Sharkovsky ordering)**.** The Sharkovsky ordering of $\mathbb{N}$ is the total order given by listing first the odd numbers greater than 1 in increasing order, then twice the odd numbers, then 2^2 times the odd numbers, and so on, finally appending the powers of 2 in decreasing order:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd \cdots \rhd 2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

The number 3 is the "largest" element of this order and 1 is the "smallest." We write $m \rhd n$ to mean that m precedes n, so m "forces" n in the sense made precise below. We write $m \lhd n$ as shorthand for $n \rhd m$; that is, m follows n, or equivalently, n forces m.

The ordering looks strange at first. The odd numbers come first (most forceful), then $2 \times$ odds, then higher powers of 2, with the pure powers of 2 being the mildest periods. One way to read it is that the ordering measures how "complex" a period is: odd periods greater than 1 are the hardest to have in isolation, while period 2 is almost unavoidable—any map with a period other than 1 must also have period 2, since 2 appears just before 1 at the end of the ordering.

A subset $\mathcal{T} \subseteq \mathbb{N}$ is a tail of the Sharkovsky order if $t \in \mathcal{T}$ and $t \rhd s$ implies $s \in \mathcal{T}$. Equivalently, $\mathcal{T}$ is a tail if and only if it has the form $\{l \in \mathbb{N} : l \lhd m\} \cup \{m\}$ for some $m \in \mathbb{N}$, or $\mathcal{T}$ is the set $\{1, 2, 4, 8, \ldots\}$ of all powers of 2, or $\mathcal{T} = \varnothing$.

The ordering enjoys the crucial doubling property

$$l \rhd r \iff 2l \rhd 2r, \tag{1}$$

which reflects that the structure of "$2^k \cdot$ odd" blocks is invariant under multiplication by 2. This identity is the combinatorial engine behind the inductive step in the proof of the forcing theorem.

4 Sharkovsky's Forcing Theorem

Theorem 7 (Sharkovsky Forcing Theorem)**.** *Let $f : I \to I$ be continuous. If m is a period for f and $m \rhd l$, then l is also a period for f.*

We sketch the elegant direct proof given by Burns and Hasselblatt [3]. The strategy is to extract from a given m-cycle a combinatorial gadget called a *Štefan sequence*, which is just sufficient to produce elementary loops of every length $l \lhd m$.

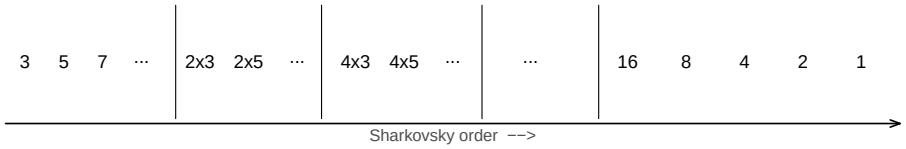

Figure 2: The Sharkovsky ordering of the natural numbers.

4.1 Cycles, coverings and the center

Let $\mathcal{O}$ be an m-cycle of f with $m \geq 2$. A closed interval whose endpoints both lie in $\mathcal{O}$ is called an $\mathcal{O}$-interval. If $J = [a, b]$ is an $\mathcal{O}$-interval then $f(a), f(b) \in \mathcal{O}$ because f permutes $\mathcal{O}$, and the IVT forces $f(J) \supseteq [\min(f(a), f(b)), \max(f(a), f(b))]$. Consequently, if I and J are both $\mathcal{O}$-intervals and J's endpoints are leftmost and rightmost points of $f(I \cap \mathcal{O})$, $I \to J$; we call such a covering $\mathcal{O}$-forced. In the rest of this section "$\to$" always means an $\mathcal{O}$-forced covering, which is what allows the argument to proceed combinatorially rather than analytically.

Let p be the rightmost point of $\mathcal{O}$ for which $f(p) > p$, and let q be the point of $\mathcal{O}$ immediately to the right of p. The center of $\mathcal{O}$ is $c := (p + q)/2$. By construction $f(p) \geq q$ and $f(q) \leq p$, so the interval $[p, q]$ covers itself and Lemma 2 gives a fixed point of f in $[p, q]$.

We say a point $x \in \mathcal{O}$ switches sides if c lies strictly between x and $f(x)$.

4.2 Štefan sequences

The following definition looks forbidding at first, but the idea is simple: a Štefan sequence is a short list of points of $\mathcal{O}$ that "straddles" the center c in a specific alternating pattern, and from which we can read off covering relations for loops of every needed length. The conditions below are exactly what is needed to make that extraction work.

Definition 8 (Burns–Hasselblatt)**.** A sequence $x_0, x_1, \ldots, x_n$ of points of $\mathcal{O}$ is a Štefan sequence if

(Š1) $\{x_0, x_1\} = \{p, q\}$;

(Š2) The sequence's points lie on alternating sides of c, and the subsequences (x_{2j}) and (x_{2j+1}) are strictly monotone, moving away from c;

(Š3) For $1 \leq j \leq n - 1$, x_j switches sides and x_{j+1} lies between c and $f(x_j)$;

(Š4) x_n does not switch sides.

Given a Štefan sequence, define $\mathcal{O}$-intervals $I_0 = [x_{n-2}, x_n]$ and, for $1 \leq j < n$, let I_j be the shortest $\mathcal{O}$-interval containing both x_0 and x_1 together with x_j. A calculation establishes the following $\mathcal{O}$-forced coverings:

$$I_1 \to I_1, \quad I_0 \to I_1, \quad I_1 \to I_2 \to \cdots \to I_{n-1} \to I_0, \quad I_0 \to I_{n-1}, I_{n-3}, \ldots$$



Figure 3: Štefan sequence example.

Proposition 9. *If the m-cycle $\mathcal{O}$ contains a Štefan sequence and $l \lhd m$, then f has an elementary $\mathcal{O}$-forced l-loop, and hence a periodic point of least period l.*

Proof sketch. Combinatorially, the three families of loops extracted from the covering graph above have lengths 1, even $l \leq n$, and anything of the form $n-1+r$ for $r \geq 1$. These lengths cover every $l \lhd m$. Each such loop avoids the interiors of the non-trivial I_j's, so by a standard criterion it is elementary, yielding a point of least period l. □

4.3 From Štefan sequences to the full theorem

The remaining step is to show that every cycle has a Štefan sequence—unless every one of its points switches sides, in which case m is necessarily even and one applies induction to the cycles $\mathcal{O}_L, \mathcal{O}_R$ of f^2 of length $m/2$. The doubling property (1) is exactly what allows this induction to close.

Thus Theorem 7 is proved.

5 Period Three and Li–Yorke Chaos

Sharkovsky's theorem has a striking corollary that predates the wide recognition of his work.

Corollary 10 (Li–Yorke, 1975)**.** *If $f : I \to I$ is continuous and has a point of period 3, then f has points of every period.*

Proof. Since $3 \rhd l$ for every positive integer $l \neq 3$ (3 is the maximum of the Sharkovsky order), Theorem 7 gives a point of period l for every l. □

Li and Yorke's 1975 paper "Period three implies chaos" [2] in fact proved something stronger than the corollary above. They showed that the existence of a period-3 point forces not only all periods, but also an *uncountable scrambled set* $S \subseteq I$ of points such that for all $x, y \in S$ with $x \neq y$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0, \quad \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

In words, pairs of orbits from S neither converge nor stay uniformly bounded apart: they are sensitively dependent on initial conditions. This is the definition that gave the word "chaos" its first rigorous mathematical meaning.

It is important to separate the two results. Sharkovsky's theorem is purely about which periods can coexist; it says nothing about sensitive dependence. Li and Yorke's theorem adds the topological information. The two together give a more complete picture of the transition to chaos in one-dimensional dynamics: once a period-3 orbit appears, all periods appear, and orbital behavior becomes chaotic in the Li–Yorke sense.

This motivates the remaining question of this paper, which in a sense runs in the opposite direction. Sharkovsky's theorem tells us what periods must be present. What if we want only a certain set of periods—can we actually build such a map? This is the converse.

6 The Converse: Constructive Proof

The goal of this section is to prove, by explicit construction, the following converse to Sharkovsky's theorem.

Theorem 11 (Converse, Elaydi [4])**.** *For every positive integer r, there exists a continuous map $f_r : I_r \to I_r$ on a closed interval $I_r \subseteq \mathbb{R}$ such that f_r has a point of prime period r but no points of any prime period s with $s \rhd r$.*

The proof proceeds by constructing f_r in three cases according to the form of r:

1. r is odd;
2. $r = 2^k(2n+1)$ with $n \geq 1$ and $k \geq 1$;
3. $r = 2^n$ is a power of 2.

In each case we give an explicit piecewise-linear map whose period structure can be verified by direct calculation. The key tool that unifies Cases 2 and Case 3 is a doubling construction: given a map with period m but not some earlier period m', one can mechanically produce a new map with periods $2m$ and $2m'$—and nothing in between. The examples below make the pattern clear before we state the general version.

6.1 Case 1: odd periods

The baseline example: period 5 but not period 3. Every odd number r can be rewritten in the form $r = 2n+1$ for some $n \in \mathbb{N}$. It is worth working through $n = 2$ in full detail before stating the general construction; the argument for general odd $r = 2n+1$ follows the same outline. Define $f : [1,5] \to [1,5]$ by

$$f(1) = 3, \quad f(2) = 5, \quad f(3) = 4, \quad f(4) = 2, \quad f(5) = 1,$$

and extend linearly on each $[i, i+1]$, for integer $1 \leq i \leq 4$.

The integers $1, 2, 3, 4, 5$ trace out a single 5-cycle under f, so 5 is a period, and each of $\{1, 2, 3, 4, 5\}$ has minimal period 5. We claim f has no point of period 3. Note first that

$$f^3([1,2]) = [2,5], \quad f^3([2,3]) = [3,5], \quad f^3([4,5]) = [1,4],$$

so none of these three intervals contains a fixed point of f^3: the only common point of each $[n, n+1]$ with its image under f^3 is a single endpoint, and at that endpoint $f^3 \neq \mathrm{id}$. The only interval that could house a period-3 point is $[3,4]$. On $[3,4]$, $f(x) = 10 - 2x$ (computed from the definition). Since f is piecewise linear, f^3 on $[3,4]$ splits into three pieces according to which linear segment $f^2(x)$ falls in:

$$f^3(x) = \begin{cases} 17 - 4x, & x \in [3,\ \frac{13}{4}], \\ 30 - 8x, & x \in [\frac{13}{4},\ \frac{7}{2}], \\ 9 - 2x, & x \in [\frac{7}{2},\ 4]. \end{cases}$$

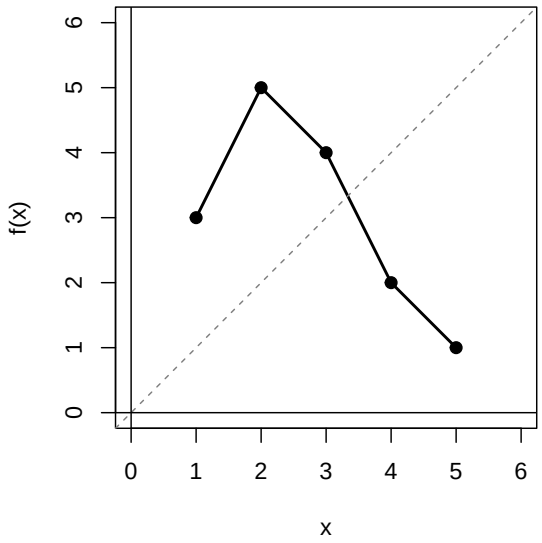

Figure 4: A continuous map on $[1, 5]$ with a point of period 5 but no point of period 3.

Setting $f^3(x) = x$ in each piece gives candidates $x = \frac{17}{5}$, $x = \frac{10}{3}$, and $x = 3$, respectively. Only $x^* = \frac{10}{3}$ lies in its corresponding subinterval $[\frac{13}{4}, \frac{7}{2}]$, so it is the unique fixed point of f^3 in $[3, 4]$. But $f(10/3) = 10 - 20/3 = 10/3$, so x^* is actually a fixed point of f, not a period-3 point. Hence f has no period-3 points.

Generalization: period $2n+1$ but not $2n-1$. For $n \geq 1$ define $f : [1, 2n+1] \to [1, 2n+1]$ by

$$\begin{aligned} &f(1) = n + 1, \quad f(2) = 2n + 1, \\ &f(k) = 2n + 3 - k \quad \text{for } 3 \leq k \leq n + 1, \\ &f(k) = 2n + 2 - k \quad \text{for } n + 2 \leq k \leq 2n + 1, \end{aligned}$$

extended linearly in between. The orbit of 1 is

$$1 \to n + 1 \to n + 2 \to n \to n + 3 \to n - 1 \to \cdots \to 2 \to 2n + 1 \to 1,$$

visiting all $2n + 1$ integers, so $2n + 1$ is a period for f.

To show there is no point of period $2n - 1$, one analyzes the behavior of f on each subinterval $[k, k + 1]$. For $[1, 2]$, iterating $2n - 1$ times yields a chain

$$[1, 2] \to [n + 1, 2n + 1] \to [1, n + 2] \to [n, 2n + 1] \to \cdots \to [2, 2n + 1],$$

so $f^{2n-1}([1, 2]) \cap [1, 2) = \emptyset$; hence $[1, 2]$ has no period-$(2n - 1)$ point. A similar argument handles every $[k, k + 1]$ except for the central interval $[n + 1, n + 2]$, which is shown separately to contain only a single fixed point.

6.2 Case 2: periods of the form $2^k(2n + 1)$

The tool here is the doubling map, which takes an example on a small interval and glues a shifted copy onto a larger interval so that every period is doubled.

From period 5 to period 10. Let $f : [1,5] \to [1,5]$ be the period-5 example from Case I. Define $g : [1,13] \to [1,13]$ by

$$g(x) = \begin{cases} f(x) + 8, & 1 \le x \le 5, \\ x - 8, & 9 \le x \le 13, \end{cases}$$

and extend linearly on $[5,9]$ by connecting $(5,9)$ to $(9,1)$ with a straight line.

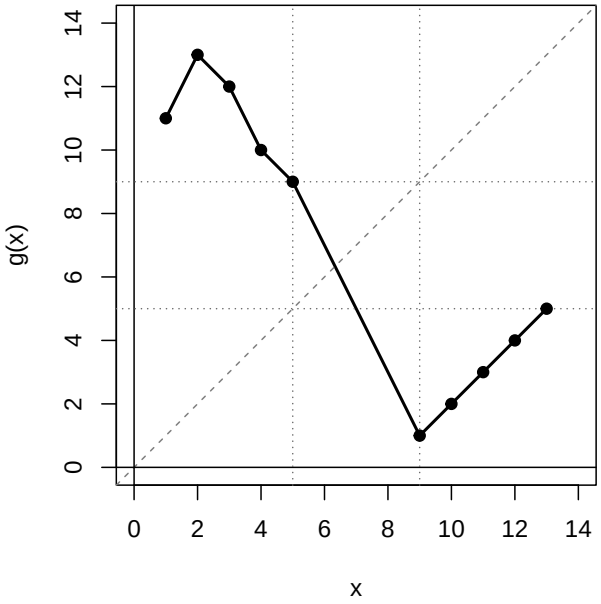

Figure 5: The doubled map g: period 10 but no period 6.

Observe that g sends $[1,5]$ into $[9,13]$ and $[9,13]$ back into $[1,5]$. Consequently for $x \in [1,5]$ we have $g^2(x) = f(x)$, so the period structure of f is inherited by g^2 and doubled for g. Since f had no period-3 points, g has no period-6 points in $[1,5] \cup [9,13]$. Other than the fixed point solution, every other point starting in $[5,9]$ is eventuallt mapped into $[1,5] \cup [9,13]$ and then trapped there. Therefore, the obly fixed point of g^6 is the fixed point of $g$itself, which is $p = 19/3$. Hence g has period 10 but not period 6.

The general doubling construction. If $f : [1, 1+h] \to [1, 1+h]$ has period $2n+1$ but not period $2n-1$, define

$$g(x) = \begin{cases} f(x) + 2h, & 1 \le x \le 1 + h, \\ x - 2h, & 1 + 2h \le x \le 1 + 3h, \end{cases}$$

on $[1, 1+3h]$, and extend linearly on $[1+h, 1+2h]$. Then g has period $2(2n+1)$ but not period $2(2n-1)$. Iterating k times yields a map with period $2^k(2n+1)$ but no period $2^k(2n-1)$.

6.3 Case 3: powers of 2

Period 2 but not 4. Let $f(x) = -x + 3$ on $[1,2]$. The unique fixed point is $3/2$, and every other point x satisfies $f(f(x)) = -(-x+3) + 3 = x$, so every non-fixed point has prime period 2. Since no point has prime period 4, this is our base example.

Period 4 but not 8. Define $f : [1, 4] \to [1, 4]$ by

$$f(1) = 3, \quad f(2) = 4, \quad f(3) = 2, \quad f(4) = 1,$$

extended linearly.

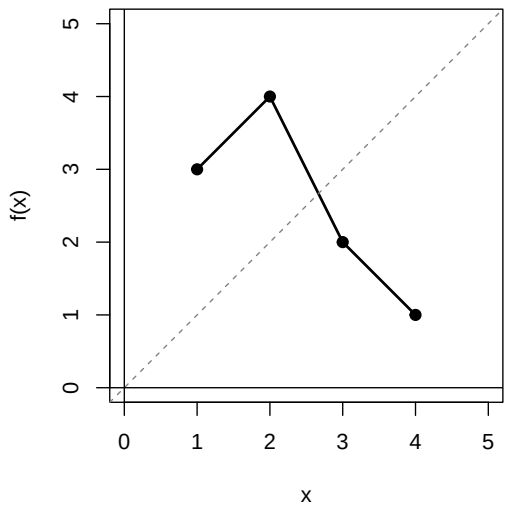

Figure 6: Period 4 but no period 8.

One checks that $f([1, 2]) = [3, 4]$ and $f([3, 4]) = [1, 2]$, with f linear on each. Therefore f^2 maps each of $[1, 2]$ and $[3, 4]$ into itself, and $f^4(x) = x$ on $[1, 2] \cup [3, 4]$. Every point in these subintervals has prime period 4 except for the points $3/2$ and $7/2$, which have period 2. On the interval $[2, 3]$, points either eventually escape into $[1, 2] \cup [3, 4]$ (and hence are attracted to an existing 2- or 4-cycle) or remain in $[2, 3]$ forever. In the latter case f acts on $[2, 3]$ as $x \mapsto -2x + 8$ and a short computation shows the only fixed point of f^8 within $[2, 3]$ is $x^* = 8/3$, which is already a fixed point of f. Consequently f has no points of period 8 or any earlier period in the Sharkovsky ordering.

General powers of 2. Applying the doubling construction of Case II to the period-4 example yields a map with period 8 but not 16; iterating produces examples for every 2^n.

6.4 Delahaye's example: exactly the powers of 2

A remarkable addendum, due to Delahaye [**?**] and included in Elaydi's paper, realizes the full set $\{2^n : n \geq 0\}$ of all powers of 2 as the period set of a single continuous map. Let $I = [0, 1]$ and

$$I_k = \left[1 - \tfrac{1}{3^k},\ 1 - \tfrac{2}{3^{k+1}}\right], \qquad k = 0, 1, 2, \ldots.$$

These are disjoint closed subintervals accumulating at 1. On each I_k place a (rescaled) copy of the Case III map having period 2^k but no period 2^{k+1}; call it f_k. Define

$$f(x) = \begin{cases} f_k(x), & x \in I_k, \\ 1, & x = 1, \end{cases}$$

and extend linearly on the gaps between consecutive I_k (joining the endpoints smoothly). The resulting f is continuous, and its set of periods is $\bigcup_k \{2^j : 0 \leq j \leq k\} = \{2^n : n \geq 0\}$. No

other periods appear: any periodic orbit that passes through a gap is eventually mapped into some I_k by the linear interpolation, and no I_k supports a cycle of period other than a power of 2 by construction. This example realizes the "powers-of-2 tail" of the Sharkovsky order—the one tail that is not of the form $\{l : l \lhd m\} \cup \{m\}$.

The idea of stacking infinitely many examples on shrinking subintervals, each contributing one more power of 2, is perhaps the neatest construction in the paper: continuity at the accumulation point $x = 1$ follows immediately from the definitions, and there is nothing else to check.

Together with the constructions of Cases I–III, the Delahaye example completes the proof of Theorem 11 and, in combination with the realization theorem alluded to in Section 4, shows that every tail of the Sharkovsky order is the period set of some continuous interval map.

7 Applications

7.1 The logistic map

The example I found for this paper is the one-parameter family

$$f_\lambda(x) = \lambda x(1 - x), \qquad x \in [0, 1],\ \lambda \in [0, 4].$$

This map is simple enough to write on one line, yet it displays the full complexity that Sharkovsky's theorem predicts.

As λ increases from 1, f_λ undergoes a cascade of period-doubling bifurcations: a period-2 orbit appears at $\lambda = 3$, then period 4, then 8, and so on, accumulating at $\lambda_\infty \approx 3.5699$ (the Feigenbaum point). For λ in this range the period set of f_λ is exactly $\{1, 2, 4, \ldots, 2^k\}$ for some finite k—the "powers-of-2 tail" that Delahaye's example realizes abstractly. What the bifurcation diagram makes viscerally clear is that this regime is genuinely non-chaotic: the orbit structure is completely predictable.

Beyond λ_∞, the picture changes. At $\lambda \approx 1 + 2\sqrt{2} \approx 3.8284$ a period-3 orbit is born, and by Sharkovsky's theorem the map instantly acquires periods of every order. The bifurcation diagram shows this as a sudden explosion of complexity after the so-called period-3 window—a transition that is sharp, not gradual.

7.2 What the theorem tells you if you observe a time series

The result has a consequence that I find practically satisfying, even from a pure-math standpoint. Suppose you are looking at a sequence of iterates $x, f(x), f^2(x), \ldots$ of some continuous one-dimensional system and you notice that the orbit appears to be periodic with period 3. Sharkovsky's theorem tells you immediately that the system also admits orbits of every other period—and by Li–Yorke's theorem, it is chaotic in the sense that there is an uncountable scrambled set of initial conditions whose orbits neither converge nor stay bounded apart.

The contrapositive is equally useful: if you observe only powers-of-2 periods, you have a certificate that you are in the non-chaotic, period-doubling regime. Together these give a clean, observation-based test for one-dimensional chaos that requires no more than identifying the periods present.

8 Conclusion

The answer to "which period sets are possible?" is clean: exactly the tails of the Sharkovsky order. What I did not expect going in was how constructive both halves of the story are. The forward

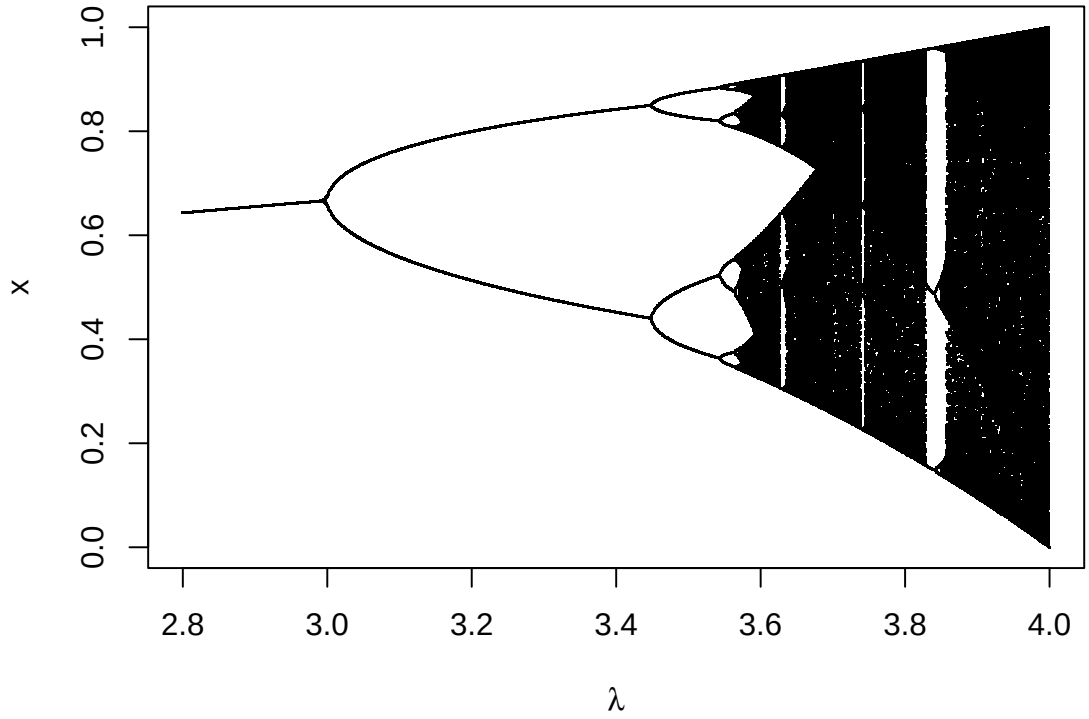

Figure 7: Bifurcation diagram of the logistic map f_λ. The period-doubling cascade is visible on the left; the period-3 window near $\lambda \approx 3.83$ triggers the onset of full chaos.

direction is essentially a covering argument—you extract loops from a given cycle and the IVT hands you periodic points of smaller period. The converse asks you to build maps from scratch, and the piecewise-linear constructions do it explicitly enough to check by running through iterates at integer points.

What both proofs share is a remarkable economy of analytic input. The forward direction of Sharkovsky's theorem—which classifies every possible period set for every continuous interval map—ultimately comes down to checking that a few differences of endpoint values have opposite signs. The converse constructions are verified the same way. The IVT does almost all the work throughout.

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