Price Dynamics with Inventory: A Second-Order ODE Analysis of Market Stability

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Abstract

This paper connects Microeconomics and Differential Equations through a model in which the market price $p(t)$ responds to demand and accumulated inventory, while inventory changes with the supply-demand gap. Starting from Evans's price-adjustment law and linear supply and demand, we derive a second-order linear ODE for price. We rewrite it as a planar system and classify the equilibrium $p_c = (D_{\max} - S_{\min})/(\alpha + \beta)$ by its eigenvalues. In every case the equilibrium is asymptotically stable; what changes is whether convergence is nonoscillatory (with possible transient overshoot) or oscillatory. We also treat the adjustment speed μ as a parameter controlling the damping and find a critical value $\mu^* = 4(\alpha + \beta)/\beta^2$ that separates the two regimes.

1 Introduction

In Microeconomics, I first met supply and demand as a *static* picture: two curves, their intersection, and a label we call equilibrium. The model tells us *where* the equilibrium price p_c sits, but not *how* the market gets there from any other starting price. MATH 225 answers that second question: given an initial condition, how does the system evolve over time, and does it converge?

Our ODE course gives us a full toolkit for this. We derive the governing equation, solve it when we can, draw trajectories in the phase plane, look at the direction field, and study how the behavior changes with the parameters. This paper applies the toolkit to the linear demand model with inventory from Xia et al. [1]. When manufacturers hold inventory, meaning the cumulative excess of supply over demand, the price no longer reacts to the current gap alone. The equation becomes second-order, which gives us damped oscillation and critically damped convergence.

I used the MATH 225 R software package to simulate the system and produce the direction fields and solution curves in this paper. I also plot the eigenvalues to see what happens when the adjustment speed changes.

2 The Economic Model

We use three building blocks from microeconomics and one dynamic pricing rule.

Definition 1 (Supply, Demand, and Inventory)**.** *Let $p(t)$ denote the market price at time $t \geq 0$. Assume $D_{\max} > S_{\min}$, so the equilibrium price is positive. The linear demand and supply functions are*

$$D(t) = D_{\max} - \beta\, p(t), \qquad \beta > 0, \tag{1}$$

$$S(t) = S_{\min} + \alpha\, p(t), \qquad \alpha > 0. \tag{2}$$

The inventory variable accumulates the excess of supply over demand:

$$H(t) = \int_0^t \big(S(\tau) - D(\tau)\big)\, d\tau. \tag{3}$$

The Evans price-adjustment law, modified to include inventory, states:

$$\frac{dp}{dt} = \mu\big(D(t) - H(t)\big), \qquad \mu > 0. \tag{4}$$

Equations (1)–(2) say the usual thing: higher prices reduce the quantity demanded and raise the quantity supplied. Equation (4) is Evans's rule with $D - H$ in place of $D - S$. When inventory is large, the price falls even if current demand is above current supply, because manufacturers can release stockpiled goods instead of raising prices. Here H measures a signed inventory imbalance: negative values represent shortages rather than literal negative stocks. We use normalized quantities and time in this simplified model so that demand and inventory can appear in the same adjustment rule.

Theorem 1 (Equilibrium Price and Inventory)**.** *The unique equilibrium of the price-inventory system is*

$$p_c = \frac{D_{\max} - S_{\min}}{\alpha + \beta}, \qquad H_c = D_{\max} - \beta p_c.$$

At this price, demand equals supply and inventory growth stops.

Proof. For inventory growth to stop, we need $S(p_c) = D(p_c)$. Setting $D_{\max} - \beta p_c = S_{\min} + \alpha p_c$ gives the stated price. For the price to stop changing as well, the adjustment law requires $D(p_c) - H_c = 0$, so $H_c = D_{\max} - \beta p_c$. □

This is the classical intersection of the supply and demand curves. What we want to know next is whether the dynamics actually drive $p(t)$ to p_c.

3 From Second-Order ODE to Standard Form

Substituting (1)–(3) into (4):

$$\frac{dp}{dt} = \mu\left(D_{\max} - \beta p(t) - \int_0^t \big(S(\tau) - D(\tau)\big) d\tau\right).$$

Differentiating both sides with respect to t and using $\frac{d}{dt}H(t) = S(t) - D(t) = (\alpha+\beta)p - (D_{\max} - S_{\min})$:

$$\frac{d^2p}{dt^2} = -\mu\beta\frac{dp}{dt} - \mu(\alpha+\beta)p + \mu(D_{\max} - S_{\min}).$$

Rearranging yields a second-order linear ODE with constant coefficients:

$$\boxed{p'' + A\,p' + B\,p \;=\; C,} \tag{5}$$

where

$$A = \mu\beta > 0, \qquad B = \mu(\alpha+\beta) > 0, \qquad C = \mu(D_{\max} - S_{\min}). \tag{6}$$

A constant particular solution satisfies $B\bar{p} = C$, which gives $\bar{p} = C/B = (D_{\max} - S_{\min})/(\alpha+\beta) = p_c$. The economic equilibrium is a constant solution of the ODE, so studying the dynamics reduces to studying the homogeneous equation.

Remark 1. *The second initial condition required by* (5) *is supplied by the original first-order law* (4)*: since* $H(0) = 0$*, we have* $p'(0) = \mu D(p_0) = \mu(D_{\max} - \beta p_0)$.

4 The Planar System and Eigenvalue Analysis

We now write (5) as a first-order planar system. Since $p_c = (D_{\max} - S_{\min})/(\alpha+\beta) = C/B$, we write $u(t) = p(t) - p_c$. Then $u' = p'$ and $u'' = p''$. Rewriting the ODE, we get

$$\begin{aligned} p'' + Ap' + Bp &= C \\ u'' + Au' + Bp - C &= 0 \\ u'' + Au' + B(p - C/B) &= 0 \\ u'' + Au' + Bu &= 0. \end{aligned}$$

Introducing $v = u'$ gives

$$\begin{aligned} u' &= v \\ v' &= -Av - Bu. \end{aligned}$$

In matrix form,

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{M}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} 0 & 1 \\ -B & -A \end{pmatrix}. \tag{7}$$

The characteristic equation $\det(\lambda I - \mathbf{M}) = 0$ is

$$\lambda^2 + A\lambda + B = 0, \tag{8}$$

with discriminant

$$\Delta \;=\; A^2 - 4B \;=\; \mu^2\beta^2 - 4\mu(\alpha+\beta). \tag{9}$$

Since $A > 0$ and $B > 0$, the product of the eigenvalues equals $B > 0$ and their sum equals $-A < 0$. If the roots are real, both are negative; if they are complex, their real part is $-A/2 < 0$. So the equilibrium $(u, v) = (0, 0)$, which corresponds to $p = p_c$ and $H = H_c$, is globally asymptotically stable. The sign of Δ only tells us the *type* of convergence.

5 Three Cases: Solutions, Direction Fields, and Long-Run Behavior

Case 1 ($\Delta > 0$) — Stable Node, Nonoscillatory Convergence

Two distinct real eigenvalues:

$$r_1 = \frac{-A - \sqrt{\Delta}}{2}, \qquad r_2 = \frac{-A + \sqrt{\Delta}}{2},$$

with $r_1 < r_2 < 0$ (both negative since their product is $B > 0$ and sum is $-A < 0$). The general solution is

$$p(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + p_c. \tag{10}$$

Both exponential terms decay, so $p(t) \to p_c$ as $t \to \infty$ without any repeated oscillations around p_c. The trajectory may still have a single transient overshoot before converging, depending on the initial velocity $p'(0)$ set by Remark 1. In the phase plane, trajectories approach the origin along a **stable node**, and generic trajectories align with the slow eigenvector (the one for r_2, which has smaller magnitude).

Economic interpretation. This case happens when the market is highly responsive (μ large) or demand is highly price-sensitive (β large). The price does not oscillate around p_c, though it may briefly overshoot before settling—the inventory that has already built up during the initial move can push the price past p_c once, but not repeatedly.

Case 2 ($\Delta = 0$) — Degenerate Node, Critically Damped

A repeated eigenvalue $r = -A/2 = -\mu\beta/2 < 0$. The general solution is

$$p(t) = \big(C_1 + C_2\, t\big) e^{-(\mu\beta/2)\, t} + p_c. \tag{11}$$

The constants can make p move away from p_c at first, but the exponential term wins and $p(t) \to p_c$ as $t \to \infty$. In the phase plane, every trajectory becomes tangent to the single eigenvector as it approaches the origin. This is the boundary case: any decrease in μ past this point switches the convergence from nonoscillatory to oscillatory.

Case 3 ($\Delta < 0$) — Stable Spiral, Underdamped Oscillation

Complex-conjugate eigenvalues $r = \sigma \pm i\omega$ with

$$\sigma = -\frac{\mu\beta}{2} < 0, \qquad \omega = \frac{\sqrt{4\mu(\alpha+\beta) - \mu^2\beta^2}}{2} > 0.$$

The general solution is

$$p(t) = e^{\sigma t}\big(C_1 \cos(\omega t) + C_2 \sin(\omega t)\big) + p_c. \tag{12}$$

The price oscillates around p_c with angular frequency ω, and the exponential envelope decays at rate $|\sigma| = \mu\beta/2$. In the phase plane, the trajectories form a **stable spiral**: the price cycles but still converges.

Economic interpretation. This case was the most interesting to me. The market is guaranteed to stabilize, and yet the price overshoots, then undershoots, then overshoots again, which is the kind of pattern we see in real commodity price cycles. The factor $e^{\sigma t}$ shrinks each swing, so the market does reach p_c in the end.

To see the three cases side by side, I used the MATH 225 R software package: `ADDDF` for the direction field and `EMODE2A` for representative trajectories. The result is Figure 1.

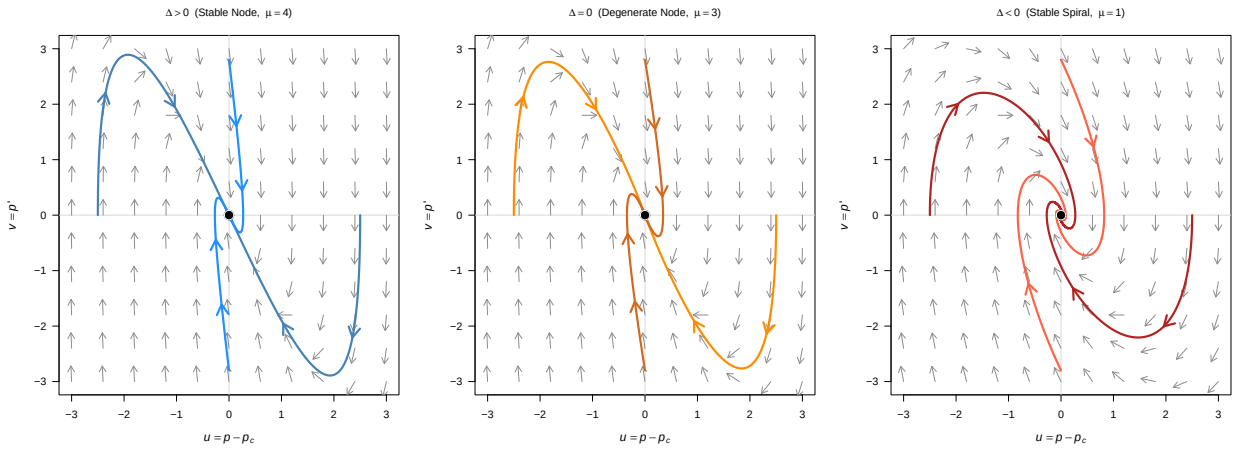

Figure 1: Direction fields and representative trajectories for system (7), drawn with `ADDDF` and `EMODE2A` from the MATH 225 R software package. From left to right, the cases are a stable node ($\mu = 4$), a critically damped node ($\mu = 3$), and a stable spiral ($\mu = 1$), with $\alpha = 1$ and $\beta = 2$. The colored curves approach the equilibrium (filled dot).

The three gray direction fields look similar, and there is a geometric reason for it. Both nullclines of system (7), the u-axis and the line $v = -((\alpha + \beta)/\beta)\, u$, do not depend on μ. So every case shares the same nullcline skeleton. Changing μ does shift arrow angles elsewhere in the plane, and that shift is most visible in Case 3 where μ is smallest. What separates the three cases is the shape of the trajectories, and that shape comes from the eigenvalue structure. The curves in Figure 1 illustrate the phase plane generally; Figure 2 uses the specific initial inventory $H(0) = 0$.

To make the differences between the three cases more concrete, I also plotted the price $p(t)$ over time in Figure 2.

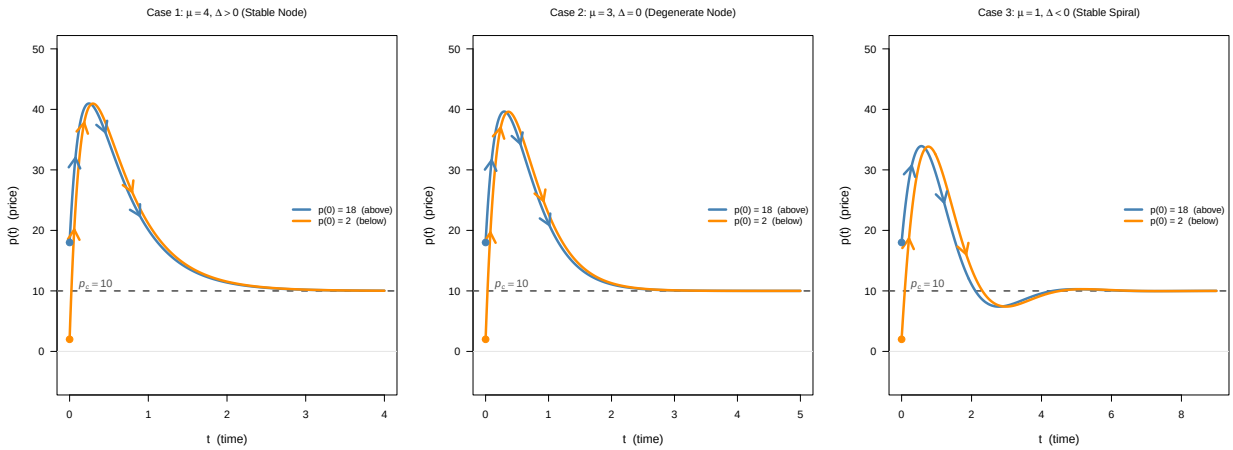

Figure 2: Price $p(t)$ over time for two initial prices, one above equilibrium ($p_0 = 18$, blue) and one below ($p_0 = 2$, orange), computed with `EMODE2A` using $\alpha = 1$, $\beta = 2$, $D_{\max} = 100$, and $S_{\min} = 70$, so $p_c = 10$ (dashed line). Since $H(0) = 0$, the initial velocity is $p'(0) = \mu(D_{\max} - \beta p_0)$, which is positive in every case: price initially rises because demand is positive and inventory starts at zero. Left ($\Delta > 0$): both curves rise and then descend to p_c from above. The orange curve crosses p_c once, while the blue curve starts above it. Center ($\Delta = 0$): the same nonoscillatory pattern at the critical value. Right ($\Delta < 0$): the price oscillates around p_c with shrinking swings before settling, which is the shape of a commodity price cycle.

6 Bifurcation Analysis: The Role of Adjustment Speed

To see how the type of convergence depends on the parameters, we take the *market adjustment speed* μ as a parameter we can change. Factoring the discriminant (9) gives

$$\Delta(\mu) = \mu\big(\mu\beta^2 - 4(\alpha + \beta)\big).$$

For fixed $\alpha, \beta > 0$, since $\mu > 0$, the sign of Δ depends only on whether μ is above the critical threshold

$$\mu^* = \frac{4(\alpha + \beta)}{\beta^2}. \tag{13}$$

Theorem 2 (Classification by Adjustment Speed)**.** *For fixed $\alpha, \beta > 0$, as μ increases through μ^*:*

- $0 < \mu < \mu^*$: $\Delta < 0$ — *stable spiral (oscillatory convergence).*
- $\mu = \mu^*$: $\Delta = 0$ — *degenerate node (critically damped).*
- $\mu > \mu^*$: $\Delta > 0$ — *stable node (nonoscillatory convergence; at most one overshoot).*

In all regimes, the price converges to the same p_c and the full system converges to the same equilibrium (p_c, H_c).

The behavior changes qualitatively across μ^*, but the destination does not. The equilibrium value p_c does not depend on μ at all, since it is set by supply and demand alone. What the transition changes is the way the market reaches p_c, not where p_c is. This is a change from a spiral to a node, not a loss of stability or the creation of another equilibrium.

In the spiral regime, the imaginary parts of the eigenvalues are $\pm\omega$, where

$$\omega(\mu) = \frac{\sqrt{4\mu(\alpha+\beta) - \mu^2\beta^2}}{2} > 0.$$

This is the oscillation frequency. It first increases, reaching its maximum when $\mu = \mu^*/2$, and then shrinks to zero as μ approaches μ^*. For $\alpha = 1$, $\beta = 2$ we have $\mu^* = 3$ and $\omega(\mu) = \sqrt{3\mu - \mu^2}$. I plotted the real and imaginary parts of both eigenvalues in Figure 3.

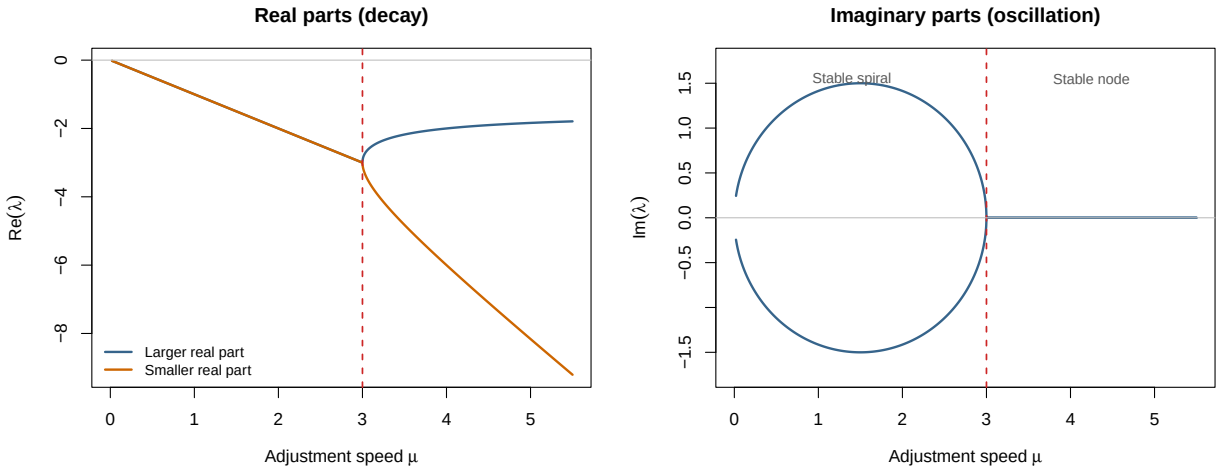

Figure 3: Eigenvalues against the adjustment speed μ, with $\alpha = 1$, $\beta = 2$, and $\mu^* = 3$ (vertical dashed line). The left panel shows the real parts of both eigenvalues. They are equal for $0 < \mu \leq 3$ and split into two negative branches for $\mu > 3$. The right panel shows the imaginary parts, which form two opposite branches in the spiral regime and are zero in the node regime. The shaded regions mark the two regimes. The equilibrium stays asymptotically stable throughout because both real parts remain negative for every $\mu > 0$.

Economic interpretation. The parameter μ measures how quickly the price reacts to imbalances. In slow-reacting markets ($\mu < \mu^*$), the difference between supply and demand builds up in inventory and can push the price past equilibrium several times, producing a damped price cycle. In fast-reacting markets ($\mu > \mu^*$), the system is over-damped and approaches equilibrium without repeated cycles. So $\mu^* = 4(\alpha+\beta)/\beta^2$ is the exact threshold at which the price cycles disappear. Since μ^* decreases as β (the price-sensitivity of demand) increases, markets with more price-sensitive demand need less adjustment speed to avoid oscillations.

7 Conclusion

The linear demand-with-inventory model lets Microeconomics and Differential Equations work on the same problem together. From the microeconomic side, we confirm what supply-and-demand theory predicts: the price converges to p_c, the intersection of the supply and demand curves. From the ODE side, we see *how* it converges. The type of convergence is set by the sign of $\Delta = \mu^2\beta^2 - 4\mu(\alpha + \beta)$, and the transition at μ^* separates oscillatory convergence from nonoscillatory convergence.

The discriminant Δ is where the two disciplines meet. The slope coefficients α and β come from the supply and demand curves, and the role of the discriminant in classifying eigenvalues is linear algebra from MATH 225. The critical threshold $\mu^* = 4(\alpha + \beta)/\beta^2$ gives us an algebraic answer to an economic question. The ODE framework does more than confirm that the market is stable: it tells us how the market gets to equilibrium, and the answer is one inequality that ties the structure of supply and demand to the speed of price signals.

References

[1] H. Xia, M. Yang, G. Han, Y. Huang, Y. Fang, and D. Meng, "Some application of ordinary differential equations in several areas of the economy," *Chaos, Solitons & Fractals*, vol. 202, article 117428, 2026.
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